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Boundary Layers and Convective Heat Transfer-Laminar; Radiation and Radiation Heat Transfer

Theme

Describes and analyzes a thermal protection system applicable to high-altitude sustained flight of hypersonic aircraft involving injection of cooled air in regions of the body wherein the maximum radiative cooling from the surface is less than the convective heating and the suction of air in regions of the body wherein the maximum radiation is normally less than the convective heating. By appropriate distribution of injection and suction an active cooling system involving a porous surface but without net energy and mass exchange is shown to be possible. Aside from the practical applicability of this system for thermal protection, the calculation of the laminar boundary layer with a distribution of mass transfer only implicitly specified by a local energy balance is of interest.

Content

The concept of a local energy balance among convective heating to the porous surface, radiation from the porous surface and energy exchange to the air in its passage through the porous surface is described first. Then there is given the exact formulation of the equations describing the laminar boundary layer for two dimensional or axisymmetric flows wherein the mass transfer distribution corresponding to that local energy balance is computed as part of the solution. An approximate method of analysis based on an eigenfunction expansion is carried out in detail. It is shown that according to this method the mass transfer distribution is given by an integro-differential equation which may be solved numerically. Several examples illustrating the effect of the several parameters entering this calculation are presented.

Analysis of an Active Thermal Protection System for High-Altitude Flight

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A thermal protection system which may be of interest for sustained, high-altitude hypersonic flight is analyzed. It involves no net mass and energy transfer, air being injected into the boundary layer in the nose region where convective heating cannot be handled by radiation and withdrawn from the boundary layer in downstream region where possible radiative cooling normally exceeds convective heating. A zero net mass transfer can be achieved. Radiation cooling plays a dominant role in the system. Quite aside from the practical applicability of the system there would appear to be interest in the analysis of the laminar boundary layer with a mass transfer distribution, which is initially unknown but which is determined by a local energy balance involving radiative transfer.

Nomenclature

f = modified stream function
g = stagnation enthalpy ratio, $h_s/h_{s,e}$
h = static enthalpy
h_s = stagnation enthalpy
j = index, 0 for two-dimensional cases; 1 for axisymmetric cases
$\tilde{m}$ = Mach number parameter, $u_e^2/2h_{s,e}$
q_r = radiative transfer from the body surface
Q_r = radiative transfer function [cf. Eq. (13)]
r = cylindrical radius
s = transformed streamwise coordinate [cf. Eq. (4)]
s_1 = value of s where $(\rho v)_w = 0$
s_2 = value of s where mass balance is achieved
S = function in radiative transfer [cf. Eq. (33)]
u = streamwise velocity component
v = normal velocity component
x = streamwise coordinate
y = normal coordinate
α = acceleration parameter $(du_e/dx)_{x=0}$
β = pressure gradient parameter
γ = ratio of specific heats
η = transformed normal coordinate [cf. Eq. (4)]
μ = viscosity coefficient
ρ = mass density

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Subscripts

$(\ )_e$ = condition external to the boundary layer
$(\ )_w$ = condition at the wall $\eta = y = 0$
$(\ )_{c,i}$ = condition in coolant chamber; $i = 1$, in injection region, $i = 2$ in suction region
$(\ )_1$ = perturbed quantities
$(\ )_{1,i}$ = perturbed quantities at $s = 0$

Introduction

FOR hypersonic cruise vehicles either in sustained flight with air-breathing propulsion or in boost-glide trajectories a significant portion of the convective heating at high altitudes can be disposed of by radiation from the exposed surfaces. However, to handle by radiation alone the convective heating at the leading edges of wings and of control surfaces and at the nose of fuselages, even with special, high-temperature materials, it may be necessary to use body radii which have excessive aerodynamic drag. Therefore at leading edge and nose regions cooling schemes, e.g., internal cooling, transpiration cooling, etc., may be necessary.‡

Of the active protection schemes which may be used in localized, critical regions transpiration cooling utilizing a coolant fluid and a porous surface is one of the most efficient from the point of view of coolant effectiveness. This is true because of the reduction in convective heating due to blockage of the high-energy external flow by the coolant as it issues from the surface and to the local absorption of the heat transmitted to the surface by the coolant in its passage through the porous material.

Transpiration cooling appears to have two disadvantages which have mitigated against its wide-spread consideration in practical designs. One is related to the material and structural problems connected with the porous material and its back-up structure, its ducting, etc. The second relates to the consumption of coolant which must be carried on the vehicle and expended during flight.

In the present work a thermal protection system is described which may be considered an extension of transpiration cooling and which may make transpiration cooling sufficiently attractive for some applications so that the development work required to overcome the first disadvantage may be undertaken. The basic notion of our system is that except for the immediate leading edge or nose region the capacity of a surface on a hypersonic vehicle at high altitudes to radiate the convective heating exceeds the normal convective heating. Thus in downstream regions we may actually apply suction, i.e., remove gas from the boundary layer and cool the withdrawn gas as it passes through the porous surface so that the internal gas temperature, i.e., the coolant temperature, is acceptably low. This withdrawn gas may be ducted forward to the leading edge or nose where radiative cooling is insufficient to handle the usual, zero-mass-transfer convective heating and where it may be injected as a transpiration coolant. In passing through the porous surface in these regions the gas, of course, becomes heated. Thus with appropriate distribution of injection and suction there can be designed a thermal protection system which is adiabatic in the sense that no internal refrigeration is required and which consumes no coolant. Only a pump to provide the requisite suction and injection across a baffle at the point of zero mass flux is required.

The basic concept of a local energy balance in such a system may be understood from the following considerations: in Fig. 1 we show schematically an element of a porous surface, the radiative and convective heating on the exposed surface, and the enthalpy change in the passage of the gas through the porous material. It can be readily shown that the condition for a steady-state energy balance is§

$$(\rho v)_w(h_w - h_c) = [\lambda(\partial T/\partial y)]_w - q_r \qquad (1)$$

Now, if the maximum radiative cooling permitted by the capabilities of the porous material is desired, then q_r and h_w should be considered independent of streamwise location. Thus in all that follows we do so. With respect to h_c two different values are considered, a higher one denoted $h_{c,1}$ in regions where injection prevails and a lower one denoted $h_{c,2}$ where suction prevails, both such that $h_{c,i} < h_w$. This is done to reflect roughly the temperature rise through the pump required to provide the pressure rise of the coolant.¶

The two terms on the right-hand side of Eq. (1) represent the convective heating and the radiative cooling. In leading edge or nose regions the former dominates and coolant must be injected i.e., make $(\rho v)_w > 0$. In downstream regions where the radiative term exceeds the convective we may withdraw gas from the boundary layer, i.e., make $(\rho v)_w < 0$. For a given body configuration, given flight conditions and specified values of exposed surface and coolant temperatures, it may be expected that in some length the integrated mass fluxes of injected gas and withdrawn gas are equal so that no net mass flux is required. In actual practice the injected and withdrawn gas may be distributed in a piecewise uniform fashion so that the problem of designing the porous surface with the continuously variable $(\rho v)_w$ required to achieve a local energy balance is simplified. Again in actual practice the section over which suction is applied may be separated by an impermeable region. In either case the basic notion of our scheme can still be retained. Because we are considering high-altitude flight, we shall assume that the boundary layer is laminar. Furthermore, our interest here is to show the main features of our thermal protection system and we therefore consider either two-dimensional or axisymmetric configurations and the idealized case of continuously distributed mass transfer.

Even with these simplifying considerations an interesting boundary layer calculation is involved. The flow is nonsimilar because we are interested in blunted wedges and blunted cones and because the mass transfer will not be distributed in accordance with similarity requirements. Although we shall set up the analysis in a form suitable for calculation by finite difference methods, our quantitative results for the distribution of mass transfer will be obtained by application of the "cold-wall" approximation and of the linearized analysis of Ref. 4. The computation effort is considerably simplified thereby while the main features of the thermal protection system are exposed. We shall, however, present some stagnation-point results based on the full equations and of interest in themselves.

Analysis

Consider the configuration, which may be two-dimensional or axisymmetric, shown schematically in Fig. 2. If we make the convenient and frequently employed assumptions that the density-viscosity product $\rho\mu \simeq \rho_e\mu_e$, and the Prandtl number are unity, then we deal with the equations[5]

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \beta[(\rho_e/\rho) - f_\eta^2] = 2s(f_\eta f_{s\eta} - f_s f_{\eta\eta}) \qquad (2)$$

$$g_{\eta\eta} + fg_\eta = 2s(f_\eta g_s - f_s g_\eta) \qquad (3)$$

‡ Reference 1 provides a review of some of the design considerations which apply to vehicles exposed to convective heating for extended periods.

§ In Ref. 2 the first author has considered the distribution of temperature in the solid and fluid for both suction and injection. Although the ideas presented there are correct, the analysis is in error due to neglect of a significant boundary condition on the interior surface.[3]

¶ The first author is indebted to J. Sternberg for pointing out the necessity of this consideration.

where the independent variables are

$$\eta \equiv \rho_e u_e r^j (2s)^{-1/2} \int_0^y (\rho/\rho_e) dy$$
$$s \equiv \int_0^x \rho_e \mu_e u_e r^{2j} dx \tag{4}$$

where the dependent variable $f(s,\eta)$ is related to the velocity components according to

$$u = u_e f_\eta \tag{5}$$

$$v = -[\rho_e\mu_e u_e r^j/\rho](2s)^{1/2}[(f/2s) + f_s + (\partial\eta/\partial x)f_\eta] \tag{6}$$

and where $\beta = \beta(s)$ is the pressure gradient parameter, taken to be a given function of the streamwise independent variable s and related to u_e according to

$$\beta \equiv (2s/u_e)(du_e/ds)$$

For present purposes it is sufficiently accurate to take an approximate equation of state $\rho \sim h^{-1}$ so that

$$(\rho_e/\rho) = (g - \tilde{m}f_\eta^2)(1 - \tilde{m})^{-1} \tag{7}$$

where $\tilde{m} = \tilde{m}(s) \equiv (u_e^2/2h_{s,e})$ is also a given function of s. Thus knowledge of $\beta(s)$ and $\tilde{m}(s)$ and Eq. (7) complete specification of the describing equations.

Boundary Conditions

We next consider the boundary and initial conditions. The former are applied at $\eta = 0$ and $\eta \to \infty$; at $\eta \to \infty$ we have the usual conditions assuring smooth joining of the boundary layer with the external flow, i.e.,

$$f_\eta(s,\infty) = g(s,\infty) = 1 \tag{8}$$

In accord with the discussion presented in the Introduction the boundary condition on the enthalpy at the wall is simple, namely

$$g(s,0) = g_w \tag{9}$$

given const. With respect to the velocity at $\eta = 0$ we impose the usual no-slip condition

$$f_\eta(s,0) = 0 \tag{10}$$

and now consider an appropriate form of Eq. (1), i.e., of the energy balance. First we note that from Eq. (6) with Eq. (10) taken into account

$$-(\rho v)_w(\rho_e\mu_e u_e r^j)^{-1} = d[(2s)^{1/2}f_w]/ds \tag{11}$$

so that an integration yields

$$(2s)^{1/2}f_w = -\int_0^s (\rho v)_w(\rho_e\mu_e u_e r^j)^{-1}ds \tag{12}$$

Furthermore, with the previously employed assumptions of $\rho\mu \simeq \rho_e\mu_e$ and of unity Prandtl number, Eq. (1) becomes

$$(\rho v)_w(g_w - g_{c,i}) = \rho_e\mu_e u_e r^j(2s)^{-1/2}g_\eta(s,0) - (q_r/h_{s,e}),\ i = 1,2$$

which may be rewritten with Eq. (11) used to eliminate $(\rho v)_w$ as

$$g_\eta(s,0) = Q_r - (2s)^{1/2}\{d[(2s)^{1/2}f_w]/ds\}(g_w - g_{c,i}) \tag{13}$$

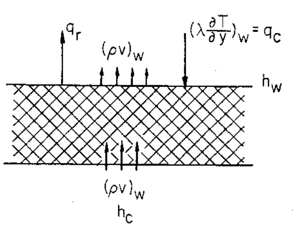

Fig. 1 The energy balance for a surface element.

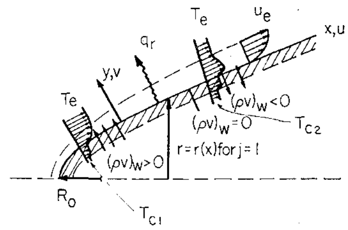

Fig. 2 Schematic representation of the flow configuration.

where

$$Q_r \equiv Q_r(s) \equiv [q_r(2s)^{1/2}(\rho_e\mu_e u_e h_{s,e}r^j)^{-1}]^*$$

With g_w and $g_{c,i}$ specified constants, Eq. (13) provides the final and important boundary condition at $\eta = 0$ and relates the convective heating, the radiative cooling, and the mass transfer.

It is appropriate to consider the requirement of no net mass transfer. Denote by s_2 the value of s corresponding to the value of x where an over-all mass balance is achieved. More precisely

$$(2\pi^j)\int_0^{x_2} (\rho v)_w r^j dx = 0$$

It then follows from Eq. (12) that

$$f_w(s_2) = 0 \tag{14}$$

Another station upstream of s_2 is also of interest, i.e., where $(\rho v)_w$ is zero; this is defined by $d[(2s)^{1/2}f_w]/ds = 0$ and denoted s_1.††

Initial Conditions

The initial conditions are not arbitrarily specified but are obtained from Eqs. (2) and (3) subject to the condition sf_s, $sg_s \to 0$ as $s \to 0$. In this case we obtain

$$f_i''' + f_if_i'' + (j+1)^{-1}(g_i - f_i'^2) = 0 \tag{15}$$

$$g_i'' + f_ig_i' = 0 \tag{16}$$

subject to the conditions

$$f_i'(0) = 0,\ f_i'(\infty) = g_i(\infty) = 1$$
$$g_i(0) = g_w,\ \text{given const}$$
$$g_i'(0) = (q_r/h_{s,e})[\rho_{s,e}\mu_{s,e}\alpha(1+j)]^{-1/2} - f_w(0)(g_w - g_{c,1})$$

where ( )′ denotes differentiation with respect to η, where the last condition derives from Eq. (13) with $s \to 0$, and where $\alpha \equiv (du_e/dx)_{x=0}$ is related to the nose radius R_0 and to flight conditions.‡‡ Note that $\beta(0) = (j+1)^{-1}$, $\tilde{m}(0) = 0$.

Stagnation-Point Solutions

Solutions to the equations yielding the initial conditions are of interest in themselves since they indicate the effect of radiative cooling on the local energy balance. These solutions involve as parameters $j = 0,1$; g_w; $g_{c,1}$; and $Q_r(0) \equiv (q_r/h_{s,e})[\rho_{s,e}\mu_{s,e}\alpha(1+j)]^{-1/2}$. Existing stagnation point solutions with mass transfer, e.g., those given in Ref. 7, do

* We note that for our particular application $Q_r > 0$ but that if q_r is interpreted as the net radiative flux at the surface, i.e., as the flux from the surface minus the incident flux from an external source, then Q_r can be negative. This is the situation described by Mirels and Welsh[6]; below we shall discuss the case of $Q_r < 0$ for the stagnation point since it may have some practical interest outside of the motivation for our study.

†† This station is of interest in connection with the location of a baffle separating the region with injection from the region with suction and of the point where $g_{c,1}$ becomes $g_{c,2}$.

‡‡ For a modified Newtonian pressure distribution

$$\alpha = R_0^{-1}(2p_{s,e}/\rho_{s,e})^{1/2}$$

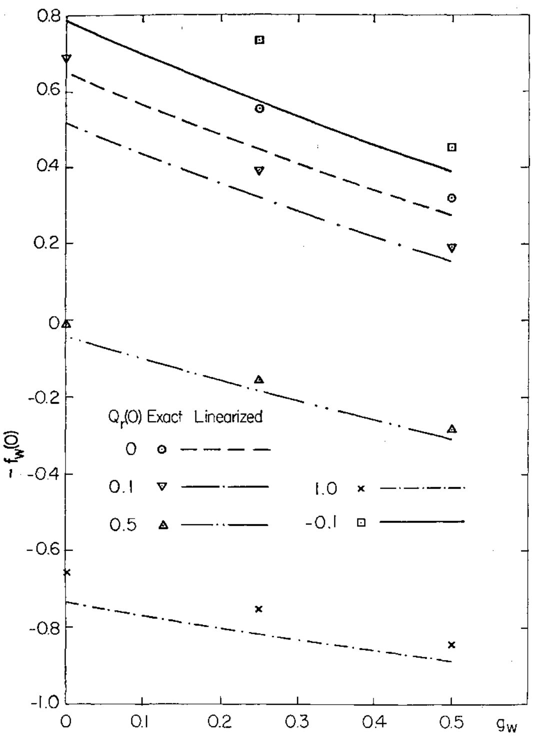

Fig. 3 Comparison of exact and linearized results for an axisymmetric stagnation point.

not explicitly include radiative cooling and an energy balance and are defined by the parameters j, g_w and $f_w(0)$. Thus these solutions may be considered to imply either a functional relation between $Q_r(0)$ and $g_{c,1}$ or for a particular value of $g_{c,1}$ to imply a particular $Q_r(0)$. However, a more useful representation results from specifications of j, g_w, g_c and $Q_r(0)$ and calculation of $f_w(0)$. For given flight conditions; nose radius, i.e., α; and permissible surface and coolant temperatures the former are known and the mass transfer rate related to $f_w(0)$ is desired. We have therefore solved Eqs. (15) and (16) by a straightforward application of quasilinearization[8] and show in Fig. 3 the results for the axisymmetric ($j = I$) case for a particular value of $g_{c,1} = 0$.§§

There are no surprises given by the results on Fig. 3. As might be expected for a fixed g_w, $-f_w(0)$ increases as the radiative flux from the surface decreases so that for a net flux into the surface more injection is required than for $Q_r(0) = 0$. On the other hand for a fixed $Q_r(0)$, $-f_w(0)$ decreases as g_w increases.

A Linearized Solution—Mass Transfer Specified

With the initial and boundary conditions, and with $\beta(s)$, $\tilde{m}(s)$ and $Q_r(s)$ specified, Eqs. (2) and (3) may be solved by existing finite difference methods modified to incorporate the wall boundary condition of Eq. (13). For present purposes the calculations would be continued downstream until the station corresponding to $(\rho v)_w = 0$, $s = s_1$, is reached. Then $g_{c,1}$ is replaced by $g_{c,2}$ and the calculation continued until the station denoted by s_2 and defined by Eq. (14) is reached. However, the main features of our thermal protection system, i.e., the distribution of mass transfer, can be exposed by a simpler analysis based on an application of Ref. 4.

To proceed we first make the convenient and frequently employed assumption that "cold wall" conditions prevail in regions where $\beta(s) \neq 0$, i.e., in the nose region where $\tilde{m} \ll 1$ and that $\beta(s) \ll 1$ where $\tilde{m} \simeq 0(1)$. Thus we drop the pressure gradient term in Eq. (2). Furthermore, we assume that the mass transfer rate is sufficiently small so that $|f_w| \ll 1$ and thus that the effect of the mass transfer may be treated in a linear fashion. These assumptions suggest that the solutions to Eqs. (2) and (3) may be approximated by

$$f(s,\eta) \simeq f_0(\eta) + f_1(s,\eta) + \ldots$$
$$g(s,\eta) \simeq g_w + (1 - g_w)f_0' + g_1(s,\eta) + \ldots \tag{17}$$

where f_0 is the Blasius solution with $f_0(0) = f_0'(0) = 0, f_0'(\infty) = 1$ and where f_1 and g_1 are obtained by substitution into Eq. (2) with $\beta(s) = 0$ and into Eq. (3) and by linearization. Thus

$$f_{1,\eta\eta\eta} + f_0 f_{1,\eta\eta} + f_0'' f_1 - 2s[f_0' f_{1,s\eta} - f_0'' f_{1,s}] \simeq 0 \tag{18}$$

$$g_{1,\eta\eta} + f_0 g_{1,\eta} - 2s f_0' g_{1,s} \simeq -f_0''(1 - g_w)(f_1 + 2s f_{1,s}) \tag{19}$$

which are now subject to the boundary conditions

$$f_1(s,0) = f_w(s), \text{ unknown}$$
$$f_{1,\eta}(s,0) = f_{1,\eta}(s,\infty) = g_1(s,0) = g_1(s,\infty) = 0$$
$$g_{1,\eta}(s,0) = -(1 - g_w)f_0'' + Q_r - (2s)^{1/2}\{d[(2s)^{1/2}f_w]/ds\}(g_w - g_{c,i})$$

The last boundary condition derives from substitution of Eqs. (17) into Eq. (13).

The initial conditions may be developed from Eqs. (15) and (16) by substitution of Eqs. (17), by neglecting the pressure gradient term and by linearization. There results

$$f_{1,i}''' + f_0 f_{1,i}'' + f_0'' f_{1,i} = 0 \tag{20}$$

$$g_{1,i}'' + f_0 g_{1,i}' = -(1 - g_w)f_0'' f_{1,i} \tag{21}$$

which are subject to the boundary conditions

$$f_{1,i}(0) = f_w(0), \text{ unknown}$$
$$f_{1,i}'(0) = g_{1,i}(0) = f_{1,i}'(\infty) = g_{1,i}(\infty) = 0$$
$$g_{1,i}(0) = -(1 - g_w)f_0'' + Q_r(0) - f_w(0)(g_w - g_{c,1})$$

We use the notation $(\ )_{1,i}$ to denote the initial conditions on the first-order corrections to the zero-order solutions.

It is convenient for exposition to consider at this point in the development that $f_w(s)$ is known but arbitrary, i.e., we drop for the time being the energy balance condition. If this is done, we can immediately write from Ref. 4 the solutions for $f_1(s,\eta)$ and $g_1(s,\eta)$; namely,

$$f_1(s,\eta) = f_w(s)\left[\hat{f}(\eta) + \sum_{n=1}^{\infty} A_n N_n(\eta)\right] - \sum_{n=1}^{\infty} (A_n\lambda_n/2)N_n(\eta)s^{-(1/2)\lambda_n} \int_0^s \xi^{(1/2)\lambda_n - 1} f_w(\xi)d\xi \tag{22}$$

$$g_1(s,\eta) = -(1 - g_w)\int_0^\infty f_0''(\eta_0)d\eta_0 \int_0^s G(s,\eta;s_0,\eta_0)(f_1 + 2s f_{1,s})\Big|_{\substack{\eta=\eta_0 \\ s=s_0}} ds_0 \tag{23}$$

§§ Although not of particular relevance to our present purposes, it is interesting to note that for $j = 1$ an "inner layer" solution applicable to large $-f_w$,[9] i.e., to large $-Q_r(0)$ and $(g_w - g_{c,1}) \ll 1$, can be found. For this solution $-f_w(0) \simeq -Q_r(g_w - g_{c,1})^{-1}$, $f_w''(0) \simeq g_w(g_w - g_{c,1})(-2Q_r)^{-1}$ so that it applies only for $Q_r < 0$. An indication of the accuracy of the estimates given by this solution is obtained for the case: $Q_r = -1$, $g_w = 0.5$, $g_{c,1} = 0$. The previous formulas give $-f_w(0) = 2$, $f_w''(0) = 0.125$; exact calculations give $-f_w(0) = 2.0002$, $f_w''(0) = 0.12512$.

where $\hat{f}(\eta)$ is defined by

$$\hat{f}''' + f_0\hat{f}'' + f_0''\hat{f} = 0, \quad \hat{f}(0) = 1, \quad \hat{f}'(0) = \hat{f}'(\infty) = 0$$

where the $N_n(\eta)$ and λ_n are eigenfunctions and eigenvalues, respectively, defined by

$$N_n''' + f_0N_n'' + \lambda_n f_0'N_n' + (1 - \lambda_n)f_0''N_n = 0$$

$$N_n(0) = N_n'(0) = N_n'(\infty) = 0$$

where $G(s,\eta;\, s_0,\eta_0)$ is a Green's function defined by

$$G(s,\eta;s_0,\eta_0) = -\sum_{m=1}^{\infty} [M_m(\eta)s^{-(1/2)\gamma_n}/2D_m] \times [M_m(\eta_0)s_0^{(1/2)\gamma_n - 1}/f_0''(\eta_0)]$$

$$D_m = \int_0^{\infty} (f_0'/f_0'')M_m^2 d\eta$$

and where M_n and γ_n are eigenfunctions and eigenvalues, respectively, defined by

$$M_m'' + f_0M_m' + \gamma_m f_0'M_m = 0, M_m(0) = M_m(\infty) = 0\text{¶¶}$$

That Eqs. (22) and (23) satisfy Eqs. (18–21) and all initial and boundary conditions except those arising from the energy balance may be verified by substitution. Now $\hat{f}(\eta)$ is known from Ref. 4 where it is denoted f_2; its important wall value is $\hat{f}''(0) = 0.72348$. In addition the first 20 eigenvalues λ_n are accurately given in Ref. 12 and the first 10 eigenvalues γ_m are given in Ref. 11. The convenient normalizations, $N_n''(0) = M_m'(0) = 1$, employed in Refs. 10–12, are retained here; this will be important when we impose the condition of energy balance. In Eqs. (22) the A_n coefficients are known, being given by integrals involving $\hat{f}$ and N_n according to

$$A_n = -C_n^{-1} \int_0^{\infty} \left(\frac{f_0'^4}{f_0''}\right)\left(\frac{N_n}{f_0'}\right)'\left(\frac{f}{f_0'}\right)' d\eta$$

where C_n is the square of the norm of $(N_n/f_0')'$, i.e.,

$$C_n = \int_0^{\infty} \left(\frac{f_0'^4}{f_0''}\right)\left(\frac{N_n}{f_0'}\right)'^2 d\eta$$

The first 20 values of A_n and C_n are given in Refs. 4 and 12, respectively.

We note that the initial solutions according to the present linearized theory are readily obtained in terms of $\hat{f}(\eta)$, again assuming that $f_w(0)$ is known. We have by comparison of Eq. (20) with the equation defining $\hat{f}(\eta)$

$$f_{1,i} = f_w(0)\hat{f}(\eta) \tag{24}$$

Furthermore, Eq. (21) can be solved to obtain

$$g_{1,i} = f_w(0)(1 - g_w)\hat{f}'(\eta) \tag{25}$$

In view of the available functions and coefficients we may consider that our solutions for $f_1(s,\eta)$, $g_1(s,\eta)$ for arbitrary $f_w(s)$ are known to an accuracy sufficient for most purposes.

Determination of the Mass Transfer Distribution

Our problem now is to determine $f_w(s)$; if Eq. (22) is substituted into Eq. (23) and the result employed in the energy balance condition, there results, after some rearrangement and integrations by parts the equation*

$$[(g_w - g_{c,i})(1 - g_w)^{-1}](2s)^{1/2}d[(2s)^{1/2}f_w]/ds = [-f_w\hat{f}''(0) - f_0''(0) + Q_r(s)(1 - g_w)^{-1}] + \Bigg\{\Bigg\{ [f_w - f_w(0)]\hat{f}''(0) - 2f_w(s)\left[\sum_{m=1}^{\infty}(E_m + \sum_{n=1}^{\infty} F_{nm})\right] + \sum_{m=1}^{\infty}\Bigg\{\gamma_m\Bigg[E_m + \sum_{n=1}^{\infty} F_{nm}(\gamma_m - 1)(\gamma_m - \lambda_n)^{-1}\Bigg] s^{-(1/2)\gamma_m}\int_0^s s_0^{(1/2)\gamma_m - 1} \times f_w(s_0)ds_0 - \sum_{n=1}^{\infty} F_{nm}\lambda_n(\lambda_n - 1)(\gamma_m - \lambda_n)^{-1}s^{-(1/2)\lambda_n} \times \int_0^s s_0^{(1/2)\lambda_n - 1} f_w(s_0)ds_0\Bigg\}\Bigg\}\Bigg\} \tag{26}$$

where the coefficients E_m, F_{nm} are defined by

$$E_m \equiv (2D_m)^{-1}\int_0^{\infty} M_m(\eta_0)\hat{f}(\eta_0)d\eta_0$$

$$F_{nm} = A_n(2D_m)^{-1}\int_0^{\infty} M_m(\eta_0)N_n(\eta_0)d\eta_0$$

Equation (26) is the basic equation for the determination (according to our linearized analysis) of the required mass transfer distribution in the form $f_w(s)$. It contains the parameters $(g_w - g_{c,i})(1 - g_w)^{-1}$, $i = 1,2$ and the forcing function $Q_r(s)(1 - g_w)^{-1}$. The solution must satisfy the condition at $s = 0$, i.e., that $s(df_w/ds)_{s=0} = 0$. In fact if we impose on the initial solutions with $f_w(0)$ specified, i.e., on Eqs. (24) and (25), the condition of energy balance, we obtain

$$-f_w(0) = \frac{f_0''(0) - Q_r(0)(1 - g_w)^{-1}}{\hat{f}''(0) + (g_w - g_{c,1})(1 - g_w)^{-1}} \tag{27}$$

We must be somewhat careful concerning the behavior of $f_w(s)$ for $s \sim 0$. Examination of Eq. (11) with the physically valid condition $(\rho v)_w \propto 1 + 0(x^2)$ for $x \ll 1$ suggests that

$$f_w(s) \simeq f_w(0) + a(2s)^{1/(j+1)} + \ldots \tag{28}$$

a behavior which clearly satisfies the requirement that $s(df_w/ds)_{s=0} = 0$. Moreover

$$Q_r(s) \simeq Q_r(0) + b(2s)^{1/(j+1)} \tag{29}$$

where b depends on the geometry of, and the external flow near, the stagnation region and may be considered known from the definition of Q_r [cf. Eq. (13)]. Substitution of Eqs. (28) and (29) into Eq. (26), and collection of powers of $2s$, i.e., of $(2s)^0$, $(2s)$, etc., yield Eq. (27) and a relation between a and b, namely,

$$a = 2Q_r(0)b(1 - g_w)^{-1}\Bigg[(g_w - g_{c,1})(1 - g_w)^{-1} + \sum_{m=1}^{\infty}\Bigg\{E_m(\gamma_m + 1)^{-1} + \sum_{n=1}^{\infty} F_{nm}[\lambda_n(\lambda_n - 1)(\gamma_m - \lambda_n)^{-1} \times (\lambda_n + 1)^{-1} - \gamma_m(\gamma_m - 1)(\gamma_m - \lambda_n)^{-1}(\gamma_m + 1)^{-1}]\Bigg\}\Bigg]^{-1} \tag{30}$$

¶¶ For a more detailed discussion of the developments leading to the eigenfunctions and eigenvalues utilized here, the reader is referred to Refs. 10 and 11. We note that our notation is slightly different from that in Ref. 4; for the present specific application, the present simpler notation is possible.

* More explicitly, note that

$$s^{-(1/2)\gamma_m}\int_0^s s_0^{(1/2)\gamma_m - (1/2)\lambda_n - 1}\left(\int_0^{s_0} \xi^{(1/2)\lambda_n - 1}f_w(\xi)d\xi\right)ds_0 = 2(\gamma_m - \lambda_n)^{-1}\left[s^{-(1/2)\lambda_n}\int_0^s s_0^{(1/2)\lambda_n - 1}f_w(s_0)ds_0 - s^{-1/2\gamma_m}\int_0^s s_0^{(1/2)\gamma_m - 1}f_w(s_0)ds_0\right]$$

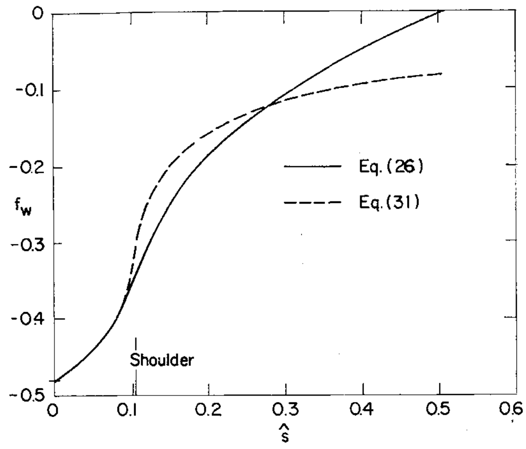

Fig. 4 Comparison of full and locally-similar solutions given by Eqs. (26) and (31), respectively; $Q_r(0) = 0.05$, $g_w = 0.2$, $g_{c,i} \equiv g_c = 0.1$.

Equations (27, 28, and 30) provide starting data for the numerical solution of Eq. (26).

In Eq. (26) the radiation term plays an important role; if $Q_r(s) \equiv 0$, then the solution to Eq. (26) is $f_w(s) \equiv f_w(0)$ which implies a special variation of $(\rho v)_w$ [cf. Eq. (11)] depending on the geometry of the body and on the variation of the external flow.† In this case, if $f_w(0) < 0$, i.e., if injection occurs at the stagnation point, there will be injection over the whole body. We thus see that radiation has a dominant effect in our protection scheme.

To clarify the interpretation of the various terms in Eq. (11) it is constructive to consider a "local similarity" solution in the sense described as follows. Suppose that

$$g \simeq g_w + (1 - g_w)f_0'$$

which would correspond to neglecting $g_1(s,\eta)$ completely, and that following Ref. 4 for small rates of mass transfer we take

$$f''(0) \simeq f_0''(0) - \hat{f}''(0)(-f_w)$$

Then

$$g'(0) = (1 - g_w)f_0''(0)[1 - \hat{f}''(0)(-f_w)]$$

For cold wall conditions with f_w = const and with small mass transfer, this would be an exact equation which substituted into the energy balance given by Eq. (13) would give an equation for $f_w(s)$, namely

$$(g_w - g_{c,i})(1 - g_w)^{-1}(2s)^{1/2}\{d[(2s)^{1/2}f_w]/ds\} = -\hat{f}''(0)f_w - f_0''(0) + Q_r(1 - g_w)^{-1} \quad (31)$$

A comparison of Eqs. (26) and (31) clearly indicates that the complex terms in {{ }} account for the deviation from local similarity in the sense implied above, i.e., they account implicitly for the downstream effect of variations of f_w. Naturally the solution of Eq. (31) is considerably simpler than that of Eq. (26); in fact from Eq. (31) we find for the case $g_{c.i} \equiv g_c$

$$-f_w(s) = \frac{f_0''(0)}{\hat{f}''(0) + (g_w - g_c)(1 - g_w)^{-1}} - \frac{s^{-n}}{2(g_w - g_c)}\int_0^s Q_r(\bar{s})\bar{s}^{n-1}d\bar{s} \quad (32)$$

† We note that if $Q_r(s) \equiv Q_r(0)$ which corresponds to an unrealistic decrease in q_r with streamwise station, then again $f_w(s) \equiv f_w(0)$. This synthetic case provides an excellent check on the numerical analysis.

where

$$n = (\tfrac{1}{2})[\hat{f}''(0) + (g_w - g_c)(1 - g_w)^{-1}]\,(1 - g_w)(g_w - g_c)^{-1}$$

The extension to the case of $g_{c,1} \neq g_{c,2}$ is straightforward, requiring revaluation of the constant of integration at $s = s_1$. As $s \sim 0$ provided $Q_r \sim Q_r(0)$, Eq. (32) yields a prediction for $f_w(0)$ in agreement with Eq. (27). As should be expected the exact and local similarity solutions start out together as $s \sim 0$.

Comparison of the Linearized and Exact Solutions for the Stagnation Point

Before proceeding to the actual determination of the mass transfer distribution $f_w(s)$ we can compare the predictions for the injection rate required for an energy balance at the stagnation point given by the linearized analysis, i.e., by Eq. (27) and by the exact solutions. We do so for $g_{c,1} = 0$ and show results for various values of $Q_r(0)$ in Fig. 3. We see that provided $Q_r(0) > 0$, i.e., provided there is a net radiative flux from the surface, the linearized results are reasonably accurate, a result which engenders confidence in the predictions of the mass transfer rates required for an energy balance downstream of the stagnation point.

Remarks on the Numerical Analysis

Space limitations prohibit discussion of the details of the numerical analysis. We thus only indicate the procedures we have used. The E_m and F_{nm} coefficients can be computed once-for-all for the available eigenfunctions and eigenvalues.‡ The integrals on the right hand side of Eq. (26) are efficiently evaluated by the adaptation of Filon's method used in Ref. 4 so that for equal increments in s, denoted Δ, they are replaced by finite sums involving the unknown values of f_w at the grid points between $s = 0$ and a particular integer multiple of Δ. Finally, the differential term on the left hand side of Eq. (26) can be approximated by a backward difference formula.

There result from this approach algebraic equations which may be solved successively for the values of f_w at increasing multiples of Δ. The solution is continued until $(\rho v)_w = 0$; then $g_{c,1}$ is replaced by $g_{c,2}$ and the calculation continued further until f_w changes sign. Interpolation then determines $s_i, i = 2$ [cf., Eq. (14)]. Either interpolation or some iterative adjustment of Δ is required to account for discontinuities in $Q_r(s)$ or of g_c.

For numerical analysis it is convenient to nondimensionalize s and to arrange $Q_r(s)$ as follows:

$$s \equiv \rho_{s,e}\mu_{s,e}h_{s,e}^{1/2}{}^{2}R_0^{2j+1}\hat{s}$$

where

$$\hat{s} \equiv \int_0^{\bar{x}} \bar{\rho}_e\bar{\mu}_e\bar{u}_e\bar{r}^{2j}d\bar{x}$$

where

$$\bar{\rho}_e \equiv \rho_e/\rho_{s,e},\ \bar{\mu}_e = \mu_e/\mu_{s,e},\ \bar{u}_e = u_e h_{s,e}^{-1/2},\ \bar{r} = r/R_0$$

and where $(\ )_{s,e}$ denotes stagnation conditions external to the boundary layer, and $\bar{x} = x/R_0$.

In terms of these nondimensional quantities

$$Q_r = \left(\frac{q_r}{\rho_{s,e}h_{s,e}^{3/2}}\right)\left(\frac{\rho_{s,e}h_{s,e}^{1/2}R_0}{\mu_{s,e}}\right)^{1/2}\frac{(2\hat{s})^{1/2}}{\bar{\rho}_e\bar{\mu}_e\bar{u}_e\bar{r}^j} = \frac{Q_r(0)S(\hat{s})}{S(0)} \quad (33)$$

where the first two factors are constants for a given set of body and flight conditions and proportional to $Q_r(0)$ and where the

‡ The values of E_m and F_{nm} for $n, m \leq 10$ are available upon your request from the first author.

third factor varies over the body and depends on body shape and is denoted $S(\hat{s})$. It is easy to change the dependent variable in Eq. (26) from s to $\hat{s}$.

For purposes of a numerical example we consider a spherically capped cone with a 20° half angle and assume that the pressure distribution is given by a modified Newtonian pressure-deflection relation; that an isentropic relation γ = constant = 1.4 relates the pressure to the other dynamic variables, in the external stream; and that $\mu_e \ \alpha \ T_e$.¶ There results for $S(\hat{s}) \equiv (2\hat{s})^{1/2}(\bar{\rho}_e\bar{\mu}_e\bar{u}_e\bar{r})^{-1}$ a distribution which has at $\hat{s} = 0.105$ a discontinuity in the slope of $S(\hat{s})$; thus we can expect a corresponding discontinuity in the second derivative of $f_w(s)$ at that station.

The change in independent variable from s to $\hat{s}$ requires Eqs. (28), (29), and (30) to be redone so that

$$f_w(\hat{s}) \simeq f_w(0) + \hat{a}(2\hat{s})^{1/2} + \ldots$$

$$Q_r(\hat{s}) \simeq Q_r(0) + \hat{b}(2\hat{s})^{1/2} + \ldots$$

while Eq. (30) stands provided a is replaced by $\hat{a}$ and b by $\hat{b}$. With the nose region and pressure distribution assumed in our numerical example

$$\hat{b} = \left(\frac{2\gamma}{\gamma - 1}\right)^{1/4}\left(\frac{5\gamma - 1}{6\gamma}\right)Q_r(0)$$

Numerical Results

We assess first the local similarity solution, i.e., we compare the predictions of Eq. (32) with that of Eq. (26) which includes explicitly the effect of the variability of f_w. Shown in Fig. 4 for typical values of the parameters determining a solution, namely $Q_r(0) = 0.05$, $g_w = 0.2$, $g_{c,i} = g_c = 0.1$, is a comparison of the two solutions. We see that the qualitative behavior of the mass transfer rate in terms of $f_w(\hat{s})$ and the quantitative behavior in the nose region $\hat{s} \lesssim 0.1$ are quite well predicted by the local similarity solution but that the quantitative behavior downstream of the nose is not sufficiently accurate to predict well the station at which $f_w(\hat{s}) = 0$, i.e., $\hat{s}_2$. The local similarity solution underpredicts the downstream cooling effect of upstream injection and therefore predicts a lower rate of permitted suction. Accordingly our further examples will be based on solutions to Eq. (26).

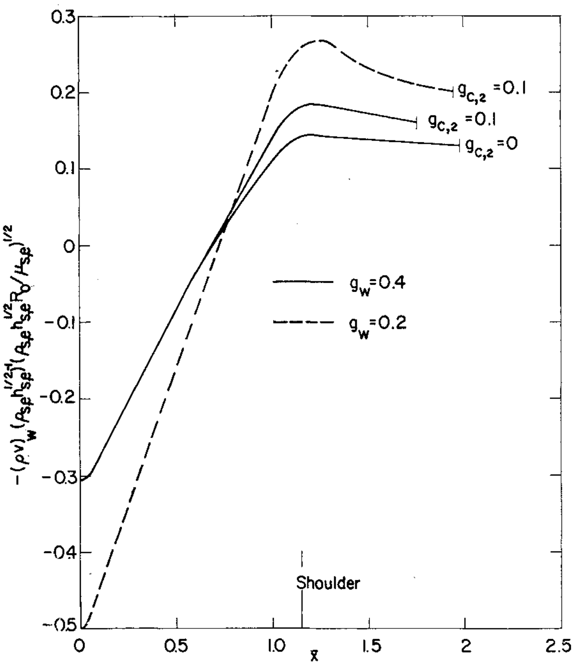

Fig. 5 Distribution of mass transfer showing effect of wall temperature and changes in coolant temperature, $g_{c,1} \neq g_{c,2}$: $Q_r(0) = 0.1$.

¶ The authors are indebted to H. Fox for assistance in making these calculations.

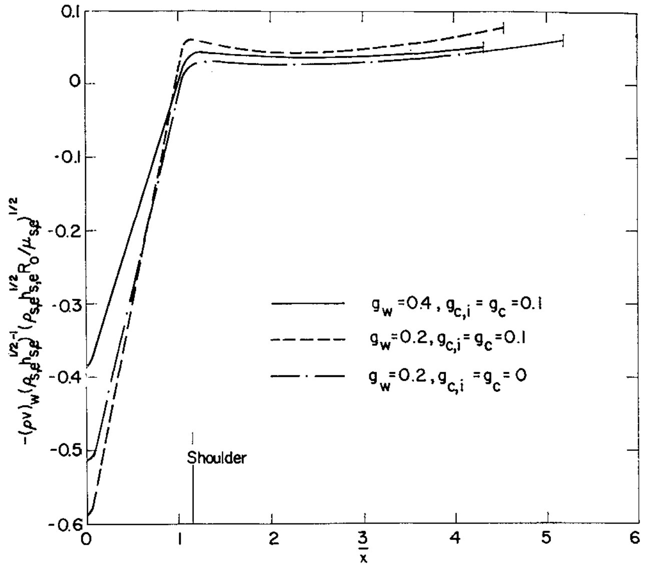

Fig. 6 Distribution of mass transfer showing effect of wall tempeature; $Q_r(0) = 0.05$.

We present next in Figs. 5 and 6 the results in terms of the nondimensional forms $-(\rho v)_w(\rho_{s,e}h_{s,e}^{1/2})^{-1}\cdot(\rho_{s,e}h_{s,e}^{1/2}R_0/\mu_{s,e})^{1/2}$ vs $\bar{x} = x/R_0$ of several sample calculations selected from many we have carried out so as to indicate the main effects of the parameters, $Q_r(0), g_w, g_{c,i}, i = 1,2$.

In Fig. 5 we show the distribution of mass transfer for two values of g_w, namely 0.4 and 0.2 for $g_{c,i} = 0.1$, $i = 1,2$ for $Q_r(0) = 0.1$. We also show the effect of $g_{c,2} < g_{c,1}$ in the case, $g_{c,1} = 0.1$, $g_{c,2} = 0$, $Q_r(0) = 0.1$.

In Fig. 6 we show the distribution of mass transfer again for two values of g_w, namely 0.4 and 0.2, again for $g_{c,i} = 0.1$, $i = 1,2$ but for a smaller radiative transfer corresponding to $Q_r(0) = 0.05$.

From these figures we can conclude, as might be anticipated from our previous remarks concerning the dominant role of the radiative transfer, the most significant parameter in determining the lengths for switching from injection to suction and for achieving zero net mass transfer is $Q_r(0)$. For a given value of $Q_r(0)$ the lower the wall temperature, i.e., the smaller g_w, the larger the injection and suction rates must be to achieve local energy balance. The reduction of $g_{c,2}$ such as may occur in practice decreases the local rate of suction and extends the length for achieving over-all mass balance. These results appear to be in qualitative agreement with intuition; our analysis, of course, provides quantitative assessments.

Concluding Remarks

We have proposed and analyzed in an approximate manner an active cooling system which may be of interest for high-altitude, hypersonic flight. The system involves no net mass or heat exchange but requires a porous surface and a pumping system for applying suction so that air is taken aboard in downstream regions and for providing injection in upstream, i.e., in nose regions. By selecting an appropriate length for given flight conditions, given radiative properties of the surface, and given body geometry, the total injected mass balances the total withdrawn fluid and zero net mass transfer can be achieved. Moreover, since each element of porous surface respects a local energy balance including radiation from the surface, there is no net heat exchange.

For a complete assessment of the applicability of our thermal protection scheme more calculations covering a variety of flight conditions and studies of practical considerations of pumping and porous surface requirements must be performed.

Quite aside from the concept of the active cooling scheme the analysis of the boundary layer with a specially constrained, a priori unknown, mass transfer distribution appears to be of interest. Clearly more accurate calculations including exact, finite difference methods can be carried out.

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